Chapter II
Existence and Continuity Theorems

In this chapter we will prove the fundamental existence theorem for ordinary differential equations, the Cauchy-Peano theorem. This local result will be extended to a global existence and uniqueness theorem under somewhat stronger conditions. Theorems about the continuous and differentiable dependence of the solutions on all the data, including parameters, will be proved. These theorems are fundamental for the qualitative study of ordinary differential equations.

Autonomous differential equations generate (local) flows. Because such flows, and in particular also semiflows, appear in other connections (e.g. partial differential equations), we will study the fundamental properties of flows in metric spaces.

The proofs are written – whenever possible – in such a way that they can be extended to the infinite dimensional case. Minor necessary modifications will be pointed out at the appropriate places. Ordinary differential equations in Banach spaces play a role in nonlinear functional analysis – in particular in connection with variational methods.

6. Preliminaries

We begin with a fundamental inequality.

(6.1) Gronwall's Lemma. *Let J be an interval in $\mathbb{R}$, $t_0 \in J$, and a, β, $u \in C(J, \mathbb{R}_+)$. If we assume that*

$$u(t) \le a(t) + \left| \int_{t_0}^{t} \beta(s) u(s)\, ds \right|, \quad \forall t \in J, \tag{1}$$

then it follows that

$$u(t) \le a(t) + \left| \int_{t_0}^{t} a(s)\beta(s) e^{\left| \int_s^t \beta(\sigma)\, d\sigma \right|}\, ds \right|, \quad \forall t \in J. \tag{2}$$

Proof. With $v(t) := \int_{t_0}^{t} \beta(s) u(s)\, ds$ it follows from (1) that

$$\dot v(t) = \beta(t) u(t) \le a(t)\beta(t) + \operatorname{sgn}(t - t_0)\beta(t) v(t), \quad \forall t \in J.$$

Multiplying this inequality by

$$\gamma(t) := \exp\left\{-\left|\int_{t_0}^{t} \beta(s)\,ds\right|\right\} = \exp\left\{-\int_{t_0}^{t} \operatorname{sgn}(s-t_0)\beta(s)\,ds\right\},$$

we obtain $\gamma\dot{v} \leq a\beta\gamma - \dot{\gamma}v$, and so $(\gamma v)^{\cdot} - a\beta\gamma \leq 0$. Now integrating and using $v(t_0) = 0$, we get:

$$\begin{aligned}\operatorname{sgn}(t-t_0)v(t) &\leq \operatorname{sgn}(t-t_0)\int_{t_0}^{t} a\beta\gamma\,ds/\gamma(t)\\ &= \left|\int_{t_0}^{t} \left[a(s)\beta(s)\gamma(s)/\gamma(t)\right]\,ds\right|, \quad \forall t \in J.\end{aligned}$$

From (1) and the definition of γ it follows that

$$\begin{aligned}u(t) &\leq a(t) + \operatorname{sgn}(t-t_0)v(t)\\ &\leq a(t) + \left|\int_{t_0}^{t} a(s)\beta(s)\exp\left\{\left|\int_{s}^{t} \beta(\sigma)\,d\sigma\right|\right\}ds\right|, \quad \forall t \in J,\end{aligned}$$

which is the estimate for u, as claimed. □

(6.2) Corollary. *Let* $a(t) = a_0(|t-t_0|)$, *where* $a_0 \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is a monotone increasing function, and assume that*

$$u(t) \leq a(t) + \left|\int_{t_0}^{t} \beta(s)u(s)\,ds\right|, \quad \forall t \in J.$$

Then we obtain the estimate

$$u(t) \leq a(t)e^{\left|\int_{t_0}^{t}\beta(s)\,ds\right|}, \quad \forall t \in J.$$

Proof. Since $a(s) \leq a(t)$ for $|s-t_0| \leq |t-t_0|$, it follows from (2) that

$$\begin{aligned}u(t) &\leq a(t)\left[1 + \left|\int_{t_0}^{t} \beta(s)\exp\{|\int_{s}^{t} \beta(\sigma)\,d\sigma|\}\,ds\right|\right]\\ &= a(t)\left[1 + \operatorname{sgn}(t-t_0)\int_{t_0}^{t} \beta(s)\exp\{\operatorname{sgn}(t-t_0)\int_{s}^{t} \beta(\sigma)\,d\sigma\}\,ds\right]\\ &= a(t)\exp\left\{\operatorname{sgn}(t-t_0)\int_{t_0}^{t} \beta(\sigma)\,d\sigma\right\}, \quad \forall t \in J,\end{aligned}$$

which is the assertion. □

Lipschitz continuous functions play an important role in the theory of (ordinary) differential equations. For this reason we want to study this class of functions more carefully and explain their relation to the continuously differentiable functions.

Assume that X and Y are metric spaces and let T be a topological space. A function $f : T\times X \to Y$ is called *uniformly Lipschitz continuous with respect to* $x \in X$, if there exists a constant $\lambda \in \mathbb{R}_+$ such that

$$d(f(t,x), f(t,\overline{x})) \leq \lambda d(x,\overline{x}), \qquad \forall x,\overline{x} \in X, \quad \forall t \in T.$$

Each $\lambda \in \mathbb{R}_+$ with this property is called a *Lipschitz constant* for f. (Of course d denotes the respective metrics in X and Y.)

The function $f : T\times X \to Y$ is called (locally) *Lipschitz continuous with respect to* $x \in X$, if every point $(t_0, x_0) \in T\times X$ has a neighborhood $U\times V$ in $T\times X$ such that $f|(U\times V)$ is uniformly Lipschitz continuous with respect to $x \in V$. Finally, we set

$$\begin{aligned} C^{0,1-}(T\times X, Y) := \{f : T\times X \to Y \mid\ & f \in C(T\times X, Y) \\ & \text{and } f \text{ is Lipschitz continuous with respect to } x \in X\}. \end{aligned}$$

If T is a single point, and therefore $f : X \to Y$, then we suppress the phrase "with respect to $x \in X$." We then set

$$C^{1-}(X,Y) := \{f : X \to Y \mid f \text{ is Lipschitz continuous}\}.$$

Of course we have

$$C^{1-}(X,Y) \subseteq C(X,Y),$$

and by definition

$$C^{0,1-}(T\times X, Y) \subseteq C(T\times X, Y).$$

Finally, if X and Y are open subsets of the Banach spaces E and F, respectively, then $C^{0,1}(T\times X, Y)$ denotes the set of all continuous functions $f : T\times X \to Y$ which have continuous partial derivatives with respect to $x \in X$. That is,

$$C^{0,1}(T\times X, Y) := \{f \in C(T\times X, Y) \mid D_2 f \in C(T\times X, \mathcal{L}(E,F))\}.$$

With this notation we obtain the following elementary but important theorem.

(6.3) Proposition. *Let E and F be Banach spaces with $D \subseteq E$ open and let T be an arbitrary topological space. Then*

$$C^{0,1}(T\times D, F) \subseteq C^{0,1-}(T\times D, F).$$

In particular, we have

$$C^{1}(D,F) \subseteq C^{1-}(D,F),$$

that is, every continuously differentiable function is Lipschitz continuous.

Proof. Let $(t_0, x_0) \in T \times D$ and $f \in C^{0,1}(T \times D, F)$ be arbitrary. Then there exists a neighborhood $U \times V$ of (t_0, x_0) in $T \times D$ such that

$$\|D_2 f(t,x) - D_2 f(t_0, x_0)\| \le 1, \quad \forall (t,x) \in U \times V.$$

With $m := 1 + \|D_2 f(t_0, x_0)\|$ we then have

$$\|D_2 f(t,x)\| \le m < \infty, \quad \forall (t,x) \in U \times V.$$

Without loss of generality we may assume that V is convex. From the mean value theorem we then obtain the estimate

$$\|f(t,x) - f(t,\overline{x})\| \le \sup_{0 \le s \le 1} \|D_2 f(t, \overline{x} + s(x - \overline{x}))\| \|x - \overline{x}\| \le m \|x - \overline{x}\|$$

for all $(t,x), (t,\overline{x}) \in U \times V$, which is our assertion. □

The following proposition has significant technical importance. In particular, it says that *every Lipschitz continuous function defined on compact subsets is uniformly Lipschitz continuous.*

(6.4) Proposition. *Let X and Y be metric spaces and let T be a compact topological space. Suppose that $K \subseteq X$ is compact and $f \in C^{0,1-}(T \times X, Y)$. Then there exists an open neighborhood W of K in X such that $f|(T \times W)$ is uniformly Lipschitz continuous with respect to $x \in W$.*

Proof. By assumption, for every $(t,x) \in T \times X$ there exists an open neighborhood $U_t \times V_x$ of (t,x) in $T \times X$ and some constant $\lambda(t,x) \in \mathbb{R}_+$ such that

$$d(f(\overline{t}, \overline{x}), f(\overline{t}, \overline{\overline{x}})) \le \lambda(t,x) d(\overline{x}, \overline{\overline{x}})$$

for all $(\overline{t}, \overline{x}), (\overline{t}, \overline{\overline{x}}) \in U_t \times V_x$. Without loss of generality we may assume that

$$V_x = \mathbb{B}(x, \epsilon(x)) := \{y \in X \mid d(y,x) < \epsilon(x)\}$$

holds for some suitable $\epsilon(x) > 0$. Since $T \times K$ is compact, there exist $(t_i, x_i) \in T \times K$, $i = 1, \ldots, m$, with

$$T \times K \subseteq \bigcup_{i=1}^{m} U_{t_i} \times \mathbb{B}(x_i, \epsilon(x_i)/2).$$

Consequently

$$W := \bigcup_{i=1}^{m} \mathbb{B}(x_i, \epsilon(x_i)/2)$$

is an open neighborhood of K in X.

First we will show that the set $f(T \times W)$ has a finite diameter. To this end, let $(t,x), (s,y) \in T \times W$ be arbitrary. Then there exist indices $i, j \in \{1, \ldots, m\}$

with $(t,x) \in U_{t_i} \times V_{x_i}$ and $(s,y) \in U_{t_j} \times V_{x_j}$. From this, from the compactness of $T \times T$, and from the continuity of $(t,s) \mapsto d(f(t,x_i), f(s,x_j))$ it follows that

$$\begin{aligned}
&d(f(t,x), f(s,y)) \\
&\le d(f(t,x), f(t,x_i)) + d(f(t,x_i), f(s,x_j)) + d(f(s,x_j), f(s,y)) \\
&\le \lambda(t_i,x_i)\epsilon(x_i) + \max_{(s,t)\in T\times T} d(f(t,x_i), f(s,x_j)) + \lambda(t_j,x_j)\epsilon(x_j) \\
&=: M_{ij} < \infty.
\end{aligned}$$

With $M := \max\{M_{ij} \mid 1 \le i,j \le m\}$ we have

$$\operatorname{diam}(f(T \times W)) \le M < \infty.$$

With

$$\delta := \min\{\epsilon(x_1), \dots, \epsilon(x_m)\}/2 > 0$$

it follows that

$$\lambda := \max\{\lambda(t_1,x_1), \dots, \lambda(t_m,x_m), \delta^{-1}\operatorname{diam}(f(T \times W))\} \in \mathbb{R}_+$$

is well defined.

Now let $(t,x),(t,y) \in T \times W$ be arbitrary. Then there is some $i \in \{1,\dots,m\}$ so that $(t,x) \in U_{t_i} \times \mathbb{B}(x_i, \epsilon(x_i)/2)$. If $d(x,y) < \delta$, then

$$d(y,x_i) \le d(y,x) + d(x,x_i) < \delta + \epsilon(x_i)/2 \le \epsilon(x_i)$$

and thus $y \in V_{x_i} = \mathbb{B}(x_i, \epsilon(x_i))$. Therefore $(t,y) \in U_{t_i} \times V_{x_i}$ and

$$d(f(t,x), f(t,y)) \le \lambda(t_i,x_i) d(x,y) \le \lambda d(x,y).$$

If, on the other hand, $d(x,y) \ge \delta$, then

$$d(f(t,x), f(t,y)) \le \operatorname{diam} f(T \times W) = [\delta^{-1}\operatorname{diam} f(T \times W)]\delta \le \lambda d(x,y).$$

Therefore $d(f(t,x), f(t,y)) \le \lambda d(x,y)$ for all $(t,x),(t,y) \in T \times W$. □

After these preparations we return again to the differential equations. For the rest of this section we make the following stipulations:

$J \subseteq \mathbb{R}$ is an open interval, E is an arbitrary Banach space (over $\mathbb{K}$), $D \subseteq E$ is open and $f \in C(J \times D, E)$.

A function $u : J_u \to D$ is called a *solution of the differential equation*

$$\dot{x} = f(t,x) \tag{3}$$

if the following holds:

(i) $J_u \subseteq J$ is a perfect interval (i.e., $\operatorname{int}(J_u) \neq \emptyset$);
(ii) $u \in C^1(J_u, D)$;
(iii) $\dot{u}(t) = f(t, u(t)), \quad \forall t \in J_u$.

If $\epsilon > 0$, then $u : J_u \longrightarrow D$ is called an *ϵ-approximate solution* of (3) if the following holds:

(i) $J_u \subseteq J$ is a perfect interval;

(ii) $u \in C(J_u, D)$, and u is piecewise continuously differentiable (i.e., J_u can be written as a finite union of perfect subintervals $I_1, \ldots, I_m$, so that u is continuously differentiable on each $\overline{I_k}$);

(iii) For every subinterval $I \subseteq J_u$ on which u is continuously differentiable we have

$$\|\dot{u}(t) - f(t, u(t))\| \le \epsilon, \quad \forall t \in I.$$

(6.5) Remarks. (a) *Let J_u be a perfect subinterval of J and let $u : J_u \longrightarrow D$. Then u is a solution of the differential equation $\dot{x} = f(t, x)$ if and only if $u \in C(J_u, D)$ and*

$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))\, ds, \quad \forall t \in J_u, \tag{4}$$

where $t_0 \in J_u$ is arbitrary.

This is an immediate consequence of the fundamental theorem of calculus. The trivial fact that the *integral equation* (4) is equivalent to the differential equation (3) is of great theoretical importance. It permits us, in fact, "to work in the space of continuous functions" without regard to differentiability questions.

(b) *Let $u : J_u \longrightarrow D$ be an ϵ-approximate solution of the differential equation $\dot{x} = f(t, x)$. Then we have*

$$\|u(t) - u(t_0) - \int_{t_0}^{t} f(s, u(s))\, ds\| \le \epsilon |t - t_0|, \quad \forall t \in J_u,$$

where $t_0 \in J_u$ is arbitrary.

Proof. We consider the case $t > t_0$. (The case $t < t_0$ is treated analogously.) There exists a decomposition $t_0 =: s_0 < s_1 < \cdots < s_m := t$ with $u|[s_i, s_{i+1}] \in C^1([s_i, s_{i+1}], D)$ for $i = 0, \ldots, m-1$. By the fundamental theorem of calculus we have

$$u(s_{i+1}) - u(s_i) = \int_{s_i}^{s_{i+1}} \dot{u}(s)\, ds.$$

From this it follows that

$$\left\| u(s_{i+1}) - u(s_i) - \int_{s_i}^{s_{i+1}} f(s, u(s))\, ds \right\| \le \int_{s_i}^{s_{i+1}} \|\dot{u}(s) - f(s, u(s))\|\, ds \le \epsilon (s_{i+1} - s_i),$$

for $i = 0, 1, \ldots, m-1$. Now the assertion follows since

$$u(t) - u(t_0) - \int_{t_0}^{t} f(s, u(s))\, ds = \sum_{i=0}^{m-1} \left[u(s_{i+1}) - u(s_i) - \int_{s_i}^{s_{i+1}} f(s, u(s))\, ds \right]. \quad \square$$

The following simple estimate will play a fundamental role later on.

(6.6) Lemma. *Let $f : J \times D \to E$ be uniformly Lipschitz continuous with respect to $x \in D$ and with Lipschitz constant λ. If $u : J_u \to D$ and $v : J_v \to D$ are, respectively, ϵ_1- and ϵ_2-approximate solutions of $\dot{x} = f(t,x)$, then for every $t_0 \in J_u \cap J_v$ we have:*

$$\|u(t) - v(t)\| \leq \{\|u(t_0) - v(t_0)\| + (\epsilon_1 + \epsilon_2)|t - t_0|\}e^{\lambda|t-t_0|},$$

for all $t \in J_u \cap J_v$.

Proof. Using (6.5 b) and the identity

$$\begin{aligned} u(t) - v(t) = &\left[u(t) - u(t_0) - \int_{t_0}^{t} f(s, u(s))\, ds\right] \\ &- \left[v(t) - v(t_0) - \int_{t_0}^{t} f(s, v(s))\, ds\right] + [u(t_0) - v(t_0)] \\ &+ \int_{t_0}^{t} [f(s, u(s)) - f(s, v(s))] ds, \end{aligned}$$

it follows that

$$\begin{aligned} \|u(t) - v(t)\| \leq &(\epsilon_1 + \epsilon_2)|t - t_0| + \|u(t_0) - v(t_0)\| \\ &+ \lambda \left| \int_{t_0}^{t} \|u(s) - v(s)\|\, ds \right| \end{aligned}$$

for all $t \in J_u \cap J_v$. Now the assertion follows from corollary (6.2). □

As a first application we prove the following *uniqueness theorem* "for Lipschitz continuous right-hand sides."

(6.7) Theorem. *Let $f \in C^{0,1-}(J \times D, E)$ and assume that $u : J_u \to D$ and $v : J_v \to D$ are solutions of $\dot{x} = f(t,x)$ with $u(t_0) = v(t_0)$, for some $t_0 \in J_u \cap J_v$. Then $u = v$ holds identically on $J_u \cap J_v$.*

Proof. It suffices to prove the assertion for every compact perfect subinterval $I \subseteq J_u \cap J_v$ with $t_0 \in I$. Since $K := u(I) \cup v(I) \subseteq D$ is compact, there exists an open neighborhood W of K in D such that $f \mid (I \times W)$ is uniformly Lipschitz continuous with respect to $x \in W$ (cf. proposition (6.4)). Now the assertion follows from lemma (6.6). □

Problems

1. Show that, under the assumptions of lemma (6.6), the estimate

$$\|u(t) - v(t)\| \leq \|u(t_0) - v(t_0)\| e^{\lambda|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{\lambda} \left[e^{\lambda|t-t_0|} - 1 \right]$$

holds for all $t, t_0 \in J_u \cap J_v$. Also show that this estimate is sharper than the one in lemma (6.6).

2. Show that the estimate in problem 1 is sharp, that is, it cannot, in general, be improved.

3. Give an example to show that the inclusions

$$C^1(D, F) \subseteq C^{1-}(D, F) \subseteq C(D, F)$$

are proper ($D \subseteq E$ is open; E, F are Banach spaces).

7. Existence Theorems

In this section we let $J \subseteq \mathbb{R}$ *be an open interval,* $E = (E, |\cdot|)$ *be a finite dimensional Banach space over* $\mathbb{K}$*,* $D \subseteq E$ *be open and* $f \in C(J \times D, E)$*.*

Moreover let $(t_0, x_0) \in J \times D$ and let the constants a, $b > 0$ be fixed so that

$$[t_0 - a, t_0 + a] \subseteq J \quad \text{and} \quad \overline{\mathbb{B}}(x_0, b) \subseteq D,$$

and set $R := [t_0 - a, t_0 + a] \times \overline{\mathbb{B}}(x_0, b)$.

(7.1) Lemma. *Let* $M := \max |f(R)|$ *and* $\alpha := \min(a, b/M)$*. Then for every* $\epsilon > 0$ *there exists an* ϵ*-approximate solution*

$$u \in C([t_0 - \alpha, t_0 + \alpha], \overline{\mathbb{B}}(x_0, b))$$

of $\dot{x} = f(t, x)$ *with* $u(t_0) = x_0$ *and*

$$|u(t) - u(s)| \leq M|t - s|, \qquad \forall t, s \in [t_0 - \alpha, t_0 + \alpha].$$

Proof. Since $f|R$ is uniformly continuous, there exists some $\delta > 0$ such that

$$|f(t, x) - f(\bar{t}, \bar{x})| \leq \epsilon, \qquad \forall (t, x), (\bar{t}, \bar{x}) \in R$$

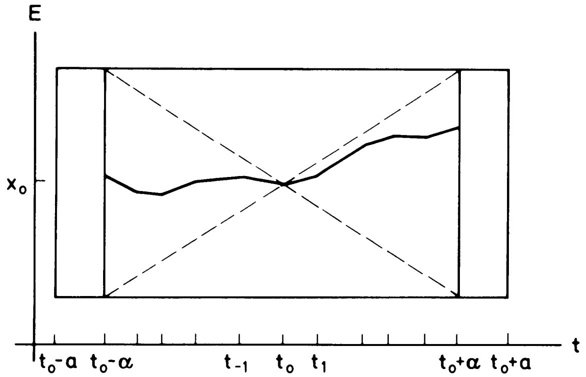
with $|t - \bar{t}| \leq \delta$ and $|x - \bar{x}| \leq \delta$. We now partition the interval $[t_0 - \alpha, t_0 + \alpha]$ into subintervals

$$t_0 - \alpha =: t_{-n} < t_{-n+1} < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_n := t_0 + \alpha,$$

such that

$$\max_{i=-n+1,\ldots,n} |t_{i-1} - t_i| \leq \min(\delta, \delta/M)$$

holds.



We then define inductively a polygonal curve, a so-called *Euler polygon*, by

$$u(t) := \begin{cases} u(t_i) + (t - t_i) f(t_i, u(t_i)), & \text{if } i \geq 0 \\ u(t_{i+1}) + (t - t_{i+1}) f(t_{i+1}, u(t_{i+1})), & \text{if } i \leq -1, \end{cases}$$

where $t_i \leq t \leq t_{i+1}$. One easily verifies that u is defined on all of $[t_0 - \alpha, t_0 + \alpha]$ and that

$$u \in C([t_0 - \alpha, t_0 + \alpha], \overline{\mathbb{B}}(x_0, b)),$$

as well as $|u(t) - u(s)| \leq M|t - s|$, holds for all $s, t \in [t_0 - \alpha, t_0 + \alpha]$. Moreover, we evidently have

$$\dot{u}(t) = f(t_i, u(t_i))$$

for all $t \in [t_i, t_{i+1}] \cap [t_0, \infty)$ and all $t \in [t_{i-1}, t_i] \cap (-\infty, t_0]$, and also

$$|u(t) - u(t_i)| \leq \delta$$

for all $t \in [t_i, t_{i+1}] \cap [t_0, \infty)$ and all $t \in [t_{i-1}, t_i] \cap (-\infty, t_0]$. From these facts it follows easily that u is an ϵ-approximate solution of $\dot{x} = f(t, x)$. □

For every $\epsilon > 0$, this lemma furnishes an ϵ-approximate solution on the fixed interval $[t_0 - \alpha, t_0 + \alpha]$. Now, if we knew that for some sequence $\epsilon_n \to 0$ the sequence of ϵ_n-approximate solutions, (u_{ϵ_n}), would converge uniformly to a function $u \in C([t_0 - \alpha, t_0 + \alpha], E)$, then a simple limit argument would show that u is a solution of the IVP $\dot{x} = f(t, x)$, $x(t_0) = x_0$. Such a convergent subsequence exists, if the set of ϵ-approximate solutions is relatively compact in $C([t_0 - \alpha, t_0 + \alpha], E)$. Hence we need a criterion for the compactness of subsets of $C([t_0 - \alpha, t_0 + \alpha], E)$. Such a criterion is furnished by the following Arzéla-Ascoli theorem. To this end, recall that for every compact topological space K and every Banach space F the space $C(K, F)$ is a Banach space with respect to the *sup-norm*

$$\|f\|_C := \max_{x \in K} \|f(x)\|.$$

(7.2) Lemma. *(Arzéla-Ascoli): Let K be a compact metric space and let F be an arbitrary Banach space. Moreover, let $\mathcal{M} \subseteq C(K, F)$. Then $\mathcal{M}$ is relatively compact (i.e., $\overline{\mathcal{M}}$ is compact) if and only if the following holds:*

(i) *$\mathcal{M}$ is equicontinuous, that is, for every $y \in K$ and every $\epsilon > 0$ there exists a neighborhood V of y in K such that*

$$\|f(x) - f(y)\| < \epsilon, \quad \forall x \in V, \ \forall f \in \mathcal{M};$$

(ii) *$\mathcal{M}(y) := \{f(y) \mid f \in \mathcal{M}\}$ is relatively compact in F for every $y \in K$.*

If F is finite dimensional, then $\mathcal{M}$ is precompact if and only if $\mathcal{M}$ is equicontinuous and bounded.

Proof. If F is an arbitrary Banach space, we refer to Lang [1] for a simple proof. Proofs which apply to more general spaces can be found, for example, in Dugundji [1] or Schubert [1].

When $\mathcal{M}$ is relatively compact, then consequently $\mathcal{M}$ is bounded. If $\mathcal{M}$ is bounded (in $C(K, F)$ of course), then evidently $\mathcal{M}(y)$ is bounded in F for every $y \in K$. Therefore $\mathcal{M}(y)$ is relatively compact when F is finite dimensional (and hence isomorphic to $\mathbb{K}^{\dim(F)}$). With this the last assertion follows from the general case. □

After these preliminaries we can now easily prove the following fundamental existence theorem.

(7.3) Theorem. *(Cauchy-Peano): Assume that $f \in C(J \times D, E)$. Then the IVP*

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

has at least one solution u on $[t_0 - \alpha, t_0 + \alpha]$ with $u([t_0 - \alpha, t_0 + \alpha]) \subseteq \overline{\mathbb{B}}(x_0, b)$.

Proof. For each $n \in \mathbb{N}^*$, lemma (7.1) implies the existence of a $\frac{1}{n}$-approximate solution u_n on $\overline{J}_\alpha := [t_0 - \alpha, t_0 + \alpha]$ such that $u_n(\overline{J}_\alpha) \subseteq \overline{\mathbb{B}}(x_0, b)$ and

$$|u_n(t) - u_n(s)| \leq M|s - t|, \quad \forall s, t \in \overline{J}_\alpha. \tag{1}$$

From (1) follows, in particular, that the set

$$\mathcal{M} := \{u_n \mid n \in \mathbb{N}^*\} \subseteq C(\overline{J}_\alpha, E)$$

is equicontinuous. Moreover, it follows from (1) that

$$|u_n(t)| \leq |u_n(t_0)| + M|t - t_0| \leq |x_0| + b,$$

for all $n \in \mathbb{N}^*$ and all $t \in \overline{J}_\alpha$. Consequently $\mathcal{M}$ is bounded in $C(\overline{J}_\alpha, E)$ and by lemma (7.2) $\mathcal{M}$ is precompact in $C(\overline{J}_\alpha, E)$. Therefore there exists some

$u \in C(\overline{J}_\alpha, E)$ and a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \to u$ in $C(\overline{J}_\alpha, E)$, as $k \to \infty$. So (u_{n_k}) converges uniformly to u on $\overline{J}_\alpha$. By (6.5 b) we have that

$$\left| u_{n_k}(t) - x_0 - \int_{t_0}^{t} f(s, u_{n_k}(s))\, ds \right| \leq \frac{1}{n_k} |t - t_0|,$$

for all $t \in \overline{J}_\alpha$ and all $k \in \mathbb{N}$. Since the convergence is uniform, it follows that we can take the limit under the integral and we get

$$u(t) - x_0 - \int_{t_0}^{t} f(s, u(s))\, ds = 0, \qquad \forall t \in \overline{J}_\alpha.$$

The assertion now follows from this and (6.5 a). □

From theorem (7.3) and theorem (6.7) we immediately obtain the following proposition.

(7.4) Local Existence and Uniqueness Theorem. *Assume that* $f \in C^{0,1-}(J \times D, E)$. *Then the IVP*

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

has a unique solution u *on* $[t_0 - \alpha, t_0 + \alpha]$.

(7.5) Remarks. (a) The solution of the IVP in theorem (7.3) is, in general, not unique, as is shown by example (5.2b).

(b) The method used in the above proof is also numerically useful. The Euler polygons can very simply be obtained by an algorithm which can easily be programmed on a computer. In general, however, only the convergence of a subsequence can be guaranteed. A better result is obtained if

$f|R$ is uniformly Lipschitz continuous with respect to $x \in \overline{\mathbb{B}}(x_0, b)$. Then the entire sequence of ϵ_n-approximate solutions, (u_{ϵ_n}), converges uniformly on $\overline{J}_\alpha := [t_0 - \alpha, t_0 + \alpha]$, as $n \to \infty$, to the unique solution u of the IVP

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0,$$

whenever $\epsilon_n \to 0$. We have the error estimate

$$|u_{\epsilon_n}(t) - u(t)| \leq \epsilon_n |t - t_0| e^{\lambda |t - t_0|},$$

where λ denotes a Lipschitz constant.

In fact, from lemma (6.6) it follows that

$$|u_{\epsilon_n}(t) - u_{\epsilon_m}(t)| \leq (\epsilon_n + \epsilon_m)\alpha e^{\lambda\alpha}, \qquad \forall t \in \overline{J}_\alpha,$$

and for all n, $m \in \mathbb{N}$. Hence (u_{ϵ_n}) is a Cauchy sequence in the Banach space $C(\overline{J}_\alpha, E)$.

(c) The above error estimate shows that the method of Euler polygons is not very well suited to numerically approximate the solution over a large time interval. One develops methods (e.g., multistep methods) in the theory of "numerical integration of ordinary differential equations" which are better suited for these purposes. □

The central result of this section is the following theorem.

(7.6) Global Existence and Uniqueness Theorem. *Assume that*

$$f \in C^{0,1-}(J \times D, E).$$

Then for every $(t_0, x_0) \in J \times D$ *there exists a unique nonextendible solution*

$$u(\cdot, t_0, x_0) : J(t_0, x_0) \to D$$

of the IVP

$$\dot{x} = f(t,x), \quad x(t_0) = x_0. \tag{2}$$

The maximal interval of existence $J(t_0, x_0)$ *is open:*

$$J(t_0, x_0) = (t^-(t_0, x_0), t^+(t_0, x_0)),$$

and we either have

$$t^- := t^-(t_0, x_0) = \inf J, \quad \textit{resp.} \quad t^+ := t^+(t_0, x_0) = \sup J,$$

or

$$\lim_{t \to t^\pm} \min \left\{ \operatorname{dist}(u(t, t_0, x_0), \partial D), |u(t, t_0, x_0)|^{-1} \right\} = 0.$$

(Here of course we mean the limit as $t \to t^-$ when $t^- > \inf J$, and $t \to t^+$ when $t^+ < \sup J$, respectively. Moreover, we use the *convention:* dist $(x, \emptyset) = \infty$.)

Proof. Let $(t_0, x_0) \in J \times D$ be fixed. By theorem (7.4) there exists some $\alpha > 0$ such that the IVP (2) has a unique solution u on $\overline{J}_\alpha := [t_0 - \alpha, t_0 + \alpha]$. Again, by theorem (7.4) there exists some $\beta > 0$ such that the IVP

$$\dot{x} = f(t,x), \quad x(t_0 + \alpha) = u(t_0 + \alpha),$$

has a unique solution v on $\overline{J}_{\alpha,\beta} := [t_0 + \alpha - \beta, t_0 + \alpha + \beta]$. Now, using theorem (6.7), we have $u = v$ on $\overline{J}_\alpha \cap \overline{J}_{\alpha,\beta}$. It then follows that the function

$$u_1 := \begin{cases} u, & \text{on } \overline{J}_\alpha \\ v, & \text{on } \overline{J}_{\alpha,\beta}, \end{cases}$$

defined on $\overline{J}_\alpha \cup \overline{J}_{\alpha,\beta}$, is a solution of the IVP (2) and is a proper extension of u. Since a similar argument can be made at $t_0 - \alpha$, we see that u can be properly extended to the right and to the left.

We now set

$$t^+ := t^+(t_0, x_0) := \sup\{\, \beta \in \mathbb{R} \mid (2) \text{ has a solution on } [t_0, \beta]\,\}$$

and

$$t^- := t^-(t_0, x_0) := \inf\{\, \gamma \in \mathbb{R} \mid (2) \text{ has a solution on } [\gamma, t_0]\,\}.$$

Then, based upon the uniqueness theorem (6.7), there exists a unique solution

$$u := u(\cdot, t_0, x_0) : J(t_0, x_0) := (t^-, t^+) \to D$$

of (2) so that u cannot be extended. In particular, it follows that $J(t_0, x_0)$ is open, because otherwise we can apply the above argument to extend u past either t^+ or t^-.

Consider the case $t^+ < \sup J$, and assume there exist some $\epsilon > 0$ and a sequence $t_i \to t^+$ such that $t_i < t^+$ and

$$|u(t_i)| \leq 1/2\epsilon \quad \text{and} \quad \operatorname{dist}(u(t_i), \partial D) \geq 2\epsilon, \qquad \forall i \in \mathbb{N}. \tag{3}$$

Without loss of generality we may assume that $\epsilon^2 \leq 1/2$. Moreover, let

$$M := \max\{|f(t,x)| \mid t_0 \leq t \leq t^+,\ |x| \leq 1/\epsilon,\ \operatorname{dist}(x, \partial D) \geq \epsilon\}$$

and $0 < \delta < \epsilon/M$. We then have

$$\begin{aligned} &|u(t_i + s)| < 1/\epsilon \text{ and } \operatorname{dist}(u(t_i + s), \partial D) > \epsilon \\ &\text{for all } i \in \mathbb{N} \text{ and } 0 \leq s \leq \min\{\delta, t^+ - t_i\}. \end{aligned} \tag{4}$$

Indeed, if (4) were false, there would exist some $k \in \mathbb{N}$ and some $\beta \in (0, \min\{\delta, t^+ - t_k\}]$ with $|u(t_k + s)| \leq 1/\epsilon$ and $\operatorname{dist}(u(t_k + s), \partial D) \geq \epsilon$ for $0 \leq s \leq \beta$ and either

$$|u(t_k + \beta)| = 1/\epsilon \quad \text{or} \quad \operatorname{dist}(u(t_k + \beta), \partial D) = \epsilon.$$

We then would have

$$|f(t_k + s, u(t_k + s))| \leq M, \qquad \text{for all } 0 \leq s \leq \beta,$$

and therefore (cf. 6.5 a)

$$|u(t_k + \beta) - u(t_k)| \leq \int_{t_k}^{t_k + \beta} |f(s, u(s))| ds \leq \beta M \leq \delta M < \epsilon.$$

Consequently, we would get

$$|u(t_k + \beta)| < |u(t_k)| + \epsilon \leq (1/2\epsilon) + \epsilon \leq 1/\epsilon,$$

since $\epsilon \leq 1/2\epsilon$ (because $\epsilon^2 \leq 1/2$), and

$$\operatorname{dist}(u(t_k + \beta), \partial D) \geq \operatorname{dist}(u(t_k), \partial D) - |u(t_k + \beta) - u(t_k)| > 2\epsilon - \epsilon = \epsilon.$$

But this contradicts the choice of β.

Because of (4) we obtain, for all $i \in \mathbb{N}$ with $t^+ - t_i \leq \delta$, the estimate

$$|u(t) - u(s)| \le \left| \int_s^t |f(\tau, u(\tau))| \, d\tau \right| \le M|t-s|, \quad \forall s, t \in [t_i, t^+). \tag{5}$$

If now (t'_k) is an arbitrary sequence such that $t'_k < t^+$ and $t'_k \to t^+$, then (5) shows that $(u(t'_k))$ is a Cauchy sequence in E. Therefore the limit

$$y := \lim_{k \to \infty} u(t'_k)$$

exists and $y \in D$, since $\operatorname{dist}(u(t), \partial D) \ge \epsilon$ for t close to t^+. It also follows from (5) that the limit

$$\lim_{k \to \infty} \int_{t_0}^{t'_k} f(s, u(s)) \, ds$$

exists. If now (s_k) is some other sequence with $s_k \to t^+$ and $s_k < t^+$, then it follows similarly that

$$\lim_{k \to \infty} u(s_k) = z \in D.$$

Hence (5) implies that

$$|y - z| = \lim_{k \to \infty} |u(t'_k) - u(s_k)| \le M \lim_{k \to \infty} |t'_k - s_k| = 0.$$

From this we obtain

$$y = \lim_{t \to t^+} u(t),$$

and a similar argument shows that

$$\lim_{t \to t^+} \int_{t_0}^{t} f(s, u(s)) ds = \int_{t_0}^{t^+} f(s, u(s)) ds,$$

i.e., the improper integral on the right converges. If we now set

$$v(t) := \begin{cases} u(t), & \text{for } t^- < t < t^+ \\ y, & \text{for } t = t^+, \end{cases}$$

we see that

$$v \in C((t^-, t^+], D)$$

and

$$v(t) = x_0 + \int_{t_0}^{t} f(s, v(s)) \, ds, \quad \forall t \in (t^-, t^+].$$

Therefore v is a solution of the IVP (2) on the interval $(t^-, t^+]$, which contradicts the choice of t^+. This shows that (3) cannot hold. We therefore have

$$\lim_{t \to t^+} \min\{\operatorname{dist}(u(t), \partial D), |u(t)|^{-1}\} = 0.$$

The argument at the point t^- is similar. □

(7.7) Corollary. *Let* $f \in C^{0,1-}(J \times D, E)$ *and*

$$\gamma^+(t_0, x_0) := \{u(t, t_0, x_0) \mid t \in [t_0, t^+(t_0, x_0))\}.$$

(a) *If* $\gamma^+(t_0, x_0)$ *is bounded, we either have* $t^+ = \sup J$ *or* $\operatorname{dist}(u(t, t_0, x_0), \partial D) \to 0$ *as* $t \to t^+$.

(b) *If* $\gamma^+(t_0, x_0)$ *is contained in a compact subset of* D*, then* $t^+ = \sup J$.

Similar assertions apply to t^- *and*

$$\gamma^-(t_0, x_0) := u\left((t^-, t_0], t_0, x_0\right).$$

The above result can be expressed somewhat imprecisely as: *either the solution exists for all time, or it approaches the boundary of* D (where the boundary of D includes the "point at infinity" ($|x| = \infty$)).

A useful criterion, implying the boundedness of all solutions of the differential equations (for finite time), is given by the following proposition. Example (5.2 a) shows that it cannot be improved significantly.

(7.8) Proposition. *Assume there exist* $\alpha, \beta \in C(J, \mathbb{R}_+) \cap L^1(J, \mathbb{R})$ *such that*

$$|f(t, x)| \leq \alpha(t)|x| + \beta(t), \qquad \forall (t, x) \in J \times D, \tag{6}$$

(i.e., f *is* <u>*linearly bounded*</u> *w.r.t.* $x \in D$*). Then every solution of* $\dot{x} = f(t, x)$ *is bounded.*

Proof. Let $u : J_u \to D$ be a solution of $\dot{x} = f(t, x)$. Then it follows from (6) and remark (6.5 a) that

$$|u(t)| \leq |u(t_0)| + \left|\int_{t_0}^{t} \beta(s)\, ds\right| + \left|\int_{t_0}^{t} \alpha(s)|u(s)|\, ds\right|, \qquad \forall t \in J_u.$$

The assertion is now a simple consequence of Gronwall's lemma (6.1). □

We obtain now easily the following fundamental *global existence and uniqueness theorem for linear differential equations* by applying the above results.

(7.9) Theorem. *Let* $A \in C(J, \mathcal{L}(E))$ *and* $b \in C(J, E)$*. Then the linear (nonhomogeneous) IVP*

$$\dot{x} = A(t)x + b(t), \qquad x(t_0) = x_0,$$

has a unique global solution for every $(t_0, x_0) \in J \times E$.

Proof. We set $f(t, x) := A(t)x + b(t)$ and choose a fixed $(s, y) \in J \times E$. Moreover, we choose some $\delta > 0$ so that $[s - \delta, s + \delta] \subseteq J$. Then for all $(t, x) \in J \times E$ with

$|t-s| \le \delta$ we have:

$$|f(s,y)-f(t,x)| \le |A(s)-A(t)||y| + \left(\max_{|\tau-s|\le\delta} |A(\tau)|\right)|x-y| + |b(s)-b(t)|.$$

This shows that $f \in C(J \times E, E)$. Also $D_2f(t,x) = A(t)$ and therefore $D_2f \in C(J \times E, \mathcal{L}(E))$. It follows that

$$f \in C^{0,1}(J \times E, E) \subseteq C^{0,1-}(J \times E, E)$$

(cf. proposition (6.3)). Finally, f is linearly bounded because

$$|f(t,x)| \le |A(t)||x| + |b(t)|, \quad \forall (t,x) \in J \times E.$$

Now the assertion follows from proposition (7.8) and theorem (7.6). □

(7.10) Remarks. (a) The Cauchy-Peano theorem is wrong if $\dim E = \infty$. For a counterexample we refer to Deimling [1]. *If a and b are chosen so small that $f|R$ is bounded, then the local existence and uniqueness theorem* (7.4) *remains also true in case the Banach space E is infinite dimensional.* (The boundedness of $f|R$ in this case can no longer be deduced from the compactness, but rather, it follows from the continuity.) From the uniform Lipschitz continuity of $f|R$ with respect to $x \in \overline{\mathbb{B}}(x_0, b)$ and from the compactness of $[t_0-a, t_0+a]$, one can easily deduce that $f|R$ is uniformly continuous. Then lemma (7.1) remains correct and the limit can be taken (without compactness), as in remark (7.5 b). For a different proof, one which is based on the historically important *Picard-Lindelöf iteration* and also applies to the infinite dimensional case, we refer to the problems at the end of this section.

(b) *In the infinite dimensional case the global existence theorem* (7.6) *remains true with the same proof if one makes the additional assumption: f is bounded on bounded subsets of D which have a positive distance from ∂D.* One should note that the latter assumption is *only* used to say something about the behavior of $u(\cdot, t_0, x_0)$ as $t \to t^{\pm}$.

(c) Based on (b), one easily verifies that theorem (7.9) remains also true in case $\dim E = \infty$.

(d) Ordinary differential equations in infinite dimensional Banach spaces play a role in some areas of *nonlinear functional analysis*. For further details in the case $\dim E = \infty$, we refer to the books by Deimling [1] and Martin [1].

Problems

1. *Banach Fixed Point Theorem.* Let X be a complete metric space and $f : X \to X$ a contraction, i.e., there exists some $\alpha \in (0,1)$ such that

$$d(f(x), f(y)) \le \alpha d(x,y), \quad \forall x, y \in X.$$